A reduction scheme for phase spaces with almost Kähler symmetry
Regularity results for momentum level sets

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Abstract. *For phase spaces with a symmetry of an almost Kähler structure, extending the symplectic structure of phase space, a scheme of reduction is proposed, in which the decomposition into level sets of a momentum mapping is supplemented by a preliminary reduction with respect to orbit type under the action of the symmetry group. The joint process of reduction is shown to be applicable for all values of the momentum mapping and any orbit type considered, without meeting any of the usual obstructions encountered in reduction. Furthermore the proposed method gives rise to reduced phase spaces or Hamiltonian systems which cannot in general be obtained by the standard process, due to Marsden and Weinstein* [8], [1], *alone.*
Applicability is demonstrated for the cotangent bundle of Riemannian manifolds, which are shown to carry an almost Kähler structure extending the canonical symplectic structure. An almost Kähler structure is constructed on the cotangent bundle for which the symmetries induced by isometries of the base manifold are almost Kähler automorphisms.

INTRODUCTION

We begin by reviewing the usual context for reduction of phase space with symmetry, due to Marsden and Weinstein [8], [1]. Given phase space as a symplectic manifold (M, ω), one considers symmetries of the symplectic structure which can

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be described in terms of the action of a Lie group G on M by symplectomorphisms:

$$\Phi : G \times M \longrightarrow M \qquad \text{with symplectomorphic restrictions}$$

$$\Phi_g : M \longrightarrow M \qquad \text{f.a. } g \in G \text{ (1)}.$$

We shall throughout assume the action to be proper cf. [1] and remark below. A momentum mapping associated with the symmetry, by definition, is a map $J : M \longrightarrow g^*$, g^* being the dual of g, the Lie algebra of G, for which the induced maps

$$\hat{J}(\xi) : M \longrightarrow \mathbb{R}$$

$$x \longmapsto \langle J(x), \xi \rangle \qquad \text{(natural pairing)}, \ \xi \in g,$$

satisfy: $\qquad i_{\xi_M}\, \omega = \mathrm{d}\hat{J}(\xi).$

Here, for $\xi \in g$, ξ_M denotes the fundamental vector field corresponding to ξ on M, [6]. In addition we assume J to be Ad*-equivariant, i.e. equivariant w.r.t. Φ on M and the coadjoint action of G on g^*, or commutativity of the diagramm:

$$\begin{array}{ccc} M & \xrightarrow{\quad J \quad} & g^* \\ {\scriptstyle \phi_g}\downarrow & & \downarrow {\scriptstyle \mathrm{Ad}^*_{g^{-1}}} \\ M & \xrightarrow{\quad J \quad} & g^* \end{array} \qquad g \in G$$

With these data the reduction lemma can be stated as follows [1]:

Let μ be a weakly regular value of J (2). If the quotient $p : J^{-1}(\mu) \longrightarrow J^{-1}(\mu)/G_\mu$ exists, then the 2-form $\tilde{\omega}$ induced according to

$$p^*\tilde{\omega} = i^*\omega \qquad \begin{array}{ccc} & & M, \omega \\ & \overset{i}{\nearrow} & \\ J^{-1}(\mu) & & \\ & \underset{p}{\searrow} & \\ & & J^{-1}(\mu)/G_\mu, \tilde{\omega} \end{array}$$

(i the inclusion map)

provides a symplectic structure for the quotient, $(J^{-1}(\mu)/G_\mu, \tilde{\omega})$ being the reduced phase space.

The relevance of the reduction scheme, and in particular the role of the momentum mapping, is best understood in application to a Hamiltonian system

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(1) Without further mention manifolds and maps are assumed to be of class C^∞ or smooth.

(2) I.e. $J^{-1}(\mu)$ is a submanifold of M and $T_xJ^{-1}(\mu) = \ker T_xJ$ f.a. $x \in J^{-1}(\mu)$.

on (M, ω) of the given symmetry, i.e. a Hamiltonian H on M which is invariant under Φ. The Hamiltonian flow F_t associated with (M, ω, H) leaves invariant the momentum level sets $J^{-1}(\mu)$ for $\mu \in g^*$. The level sets themselves, however, are not in general symplectic submanifolds of M in the sense that $i^*\omega$ may be degenerate. By virtue of Ad*-equivariance, there is on the other hand the residual symmetry described by the induced action of G_μ (the isotropy group of μ under the coadjoint action) on $J^{-1}(\mu)$. In the case stated this symmetry is factored out to give rise to a symplectic manifold. Nondegeneracy of $\tilde{\omega}$ is assured by weak regularity of μ, [1]. The whole process extends naturally to Hamiltonian systems, in that H on M induces a reduced Hamiltonian $\tilde{H}$ on $J^{-1}(\mu)/G_\mu$ by the requirement $\tilde{H} \circ p = H \circ i$. The reduced Hamiltonian system $(J^{-1}(\mu)/G_\mu, \tilde{\omega}, \tilde{H})$ represents the dynamics of the original one, for momentum μ, condensed w.r.t. the given symmetry. Both the analysis of Hamiltonian systems with symmetry by means of decomposition (e.g. Jacobi's elimination of the node, [1]) and the possibility to generate nontrivial Hamiltonian systems or even just symplectic manifolds from known or well understood systems with symmetry by means of reduction, account for the interest in a reduction scheme. The latter point has been successfully applied in the search for and understanding of integrable systems, [9].

The above statement of the reduction lemma points at obstructions that may occur in the process of reduction; the crucial points, which of course are interrelated, are

- regularity of $J^{-1}(\mu)$
- existence of the quotient manifold
- nondegeneracy of the induced form.

As regards the first and most relevant point Sard's theorem is usually invoked to guarantee regularity, and hence weak regularity, for almost all μ in g^*. For a general treatment, however, this would seem less satisfactory, since little is known about the nature of $J(M)$ as a topological subspace of g^*.

On the other hand, the defining property of an Ad*-equivariant momentum mapping suggests strong links between the local structure of $J^{-1}(\mu)$ and the local orbit structure under the action of G. In an approach based on additional information about the action of the symmetry group one might therefore expect to take care of obstructions implicitly within a suitably adapted scheme of reduction. Particularly detailed knowledge of the orbit structure is available in the case of actions by isometries of a Riemannian structure. With a view to reduction it seems natural to consider almost Kähler symmetries, i.e. actions by automorphisms of an almost Kähler structure. The presence of symplectic and Riemannian structure intertwined by means of an almost complex structure proves exceedingly useful in establishing regularity criteria. In the same context

– making use of the properties of isometric actions – Arms, Marsden and Moncrief [2] established the conical nature of the singularities occuring in the zero momentum level. We shall here use largely analogous methods to derive specific regularity results, which are independent of the momentum value considered and which will ultimately allow for a process of reduction for almost Kähler symmetric systems meeting none of the indicated obstructions.

DEFINITIONS AND PRELIMINARIES

An almost Kähler manifold (M, ω, I, g) is a manifold M with an almost complex structure I, and an I-invariant (i.e. hermitian) metric g, whose fundamental 2-form ω defined by

$$\omega_x(X, Y) = g_x(I_x X, Y) \qquad \text{for} \qquad X, Y \in T_x M$$

is closed and hence a symplectic form for M; for an introduction see [6]. (If in addition the Nijenhuis torsion of I vanishes, I is complex and a Kähler manifold results [3], [6]). The automorphisms of an almost Kähler structure are diffeomorphisms of M which at the same time are symplectomorphisms, almost complex maps and isometries w.r.t ω, I and g. It follows from the definitions that any two of these conditions imply the third.

An important tool in our analysis of the local orbit structure under an action $\Phi : G \times M \longrightarrow M$ will be the notion of a G-slice [10] at a point $x \in M$, S_x say, defined by the following conditions:

(1) S_x is a closed submanifold of $G.S_x$ containing x, $G.S_x$ open in M
(2) The isotropy group $I(x, G) = G_x$ at x leaves S_x invariant: $G_x.S_x \subset S_x$
(3) for $g \notin G_x$: $S_x \cap \Phi_g(S_x) = \phi$.

From the definition it is seen that S_x locally represents M 'modulo the action of G', which is put more clearly in the following

LEMMA (1). *Let S be a G-slice for the proper action $\Phi : G \times M \longrightarrow M$ at $x_0 \in M$. Then $N := \bigcap_{x \in S} I(x, G)$ is a normal subgroup of $H := I(x_0, G)$; $G.S$ is diffeomorphic to a fibre bundle with standard fibre S and group H/N associated with the principal fibre bundle G/N over G/H (where the action of H/N on S is naturally induced by Φ). Moreover the natural diffeomorphism is equivariant w.r.t. the obvious actions of G.*

Using a local trivialization of the bundle, there is in particular an equivariant local diffeomorphism Ψ of a neighbourhood V of x_0 in M and $S \times U$, U a neighbourhood of eH in G/H:

$$\begin{array}{ccc} S\times U & \xrightarrow{\quad\psi\quad} & V \\ {\scriptstyle \mathrm{id}\times\varphi_g|}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi_g|} \\ S\times U & \xrightarrow{\quad\psi\quad} & V \end{array}$$

commuting where defined,

where $\varphi_g|$ and $\Phi_g|$ are suitable restrictions of the natural action $\varphi : G \times G/H \longrightarrow G/H$ and Φ, rsp. It follows that in a neighbourhood of a point which admits a G-slice all isotropy groups are conjugate to subgroups of the isotropy group at x.

Dealing with proper actions by isometries, there is a natural choice for a G-slice at a point $x_0 \in M$, namely the image under the exponential map of a suitable ball in $(T_{x_0} G.\, x_0)^{\perp}$, the orthogonal complement in $T_{x_0}M$ of the tangent space to the orbit of x_0. Following [2] we shall reserve the name affine G-slice for this construction.

As regards a detailed introduction to the concept of G-slices and the discussion of their existence in a general topological context, we refer the reader to Palais [10], who originally developed the notion for a topological rather than differential theory and for the action of compact groups. For compactness of the group we here substitute a) a proper action which gives rise to orbits which are submanifolds locally [1], and b) the isometry property which – just as compactness would – allows to choose a tubular neighbourhood of the orbit of a point in a uniform way in compliance with (3). See [4]. From these ingredients a proof for the existence of affine G-slices as well as of lemma (1) is easily put together.

A first and crucial characterization of the local orbit structure involves the determination of the isotropy groups of points. Following corresponding notions in [10] we define

- the symmetry type of $x \in M$ to be the identity component of the isotropy group at $x : \hat{I}(x, G)$
- the orbit type of $x \in M$ to be the equivalence class $(\hat{I}(x, G)) := \{H \subset G/H \sim \hat{I}(x, G)\}$,

i.e. the symmetry type gives the local structure of the isotropy group while the orbit type specifies this group up to conjugation, as occurs along the orbit of x under G. Using affine G-slices and lemma (1) the following is established:

LEMMA (2). *For a proper action by isometries on M and any (closed and connected) subgroup H of G the sets*

$$M_H := \{x \in M / \hat{I}(x, G) = H\} \qquad \textit{and}$$

$$M_{(H)} := \{x \in M / \hat{I}(x, G) \sim H\}$$

are embedded submanifolds of M (if not empty).

We refer to them as submanifolds of constant symmetry and orbit type, resp.; (The same holds for any proper action allowing of G-slices at every point).

THE PROPOSED SCHEME OF REDUCTION

We shall consider the following data fixed throughout:

(M, ω, I, g) an almost Kähler manifold;

$\Phi : G \times M \longrightarrow M$ a proper action by almost Kähler automorphisms;

$J : M \longrightarrow \mathfrak{g}^*$ an Ad*-equivariant momentum mapping associated with Φ;

(and at places: H any Hamiltonian invariant under Φ).

The first step of the intended process of reduction will consist of a decomposition w.r.t. local orbit structure.

To motivate this step, we first observe that the flow of H, F_t say, commutes with the action of the group:

$$\begin{array}{ccc} M & \xrightarrow{\quad F_t \quad} & M \\ {\scriptstyle \phi_g}\downarrow & & \downarrow{\scriptstyle \phi_g} \\ M & \xrightarrow{\quad F_t \quad} & M \end{array}$$

commutes wherever defined.

It follows that both symmetry and orbit type are preserved along the Hamiltonian orbits, or that both M_H and $M_{(H)}$ are invariant under the Hamiltonian flow for any subgroup H of G.

The situation is however as follows:

• M_H, as shown in [2] and below, is a symplectic submanifold of M, i.e. to say that for $i : M_H \longrightarrow M$, the inclusion, $i^*\omega$ is nondegenerate; but M_H is not invariant under the full action of G and for this reason the restriction of J to M_H cannot be used along the lines of the reduction lemma for further reduction.

• $M_{(H)}$ on the other hand is G-invariant but not in general a symplectic submanifold in the above sense (see below for a necessary and sufficient criterion).

The aim is to show that nonetheless a process of reduction analogous to the procedure of the reduction lemma can be applied to the submanifolds $M_{(H)}$ and leads to reduced symplectic or Hamiltonian structures. The point is that this procedure – constituting the second step of the proposed reduction scheme – meets none of the usual obstructions when applied in restriction to each of the $M_{(H)}$. Or to put it another way that the singularities which do occur in the momentum level sets in M are absorbed in the process of reduction w.r.t. orbit type. An indication for this phenomenon can already be found in [2] for the zero level. This will be our central result, stated as a theorem below. The proof is organized in three main steps:

(I) $J^{-1}(\mu) \cap M_H$ is an embedded submanifold of M,

(II) $J^{-1}(\mu) \cap M_{(H)}$ is an embedded submanifold,

(III) ω induces a symplectic form on the quotient $J^{-1}(\mu) \cap M_{(H)}/G_\mu$.

As (I) is concerned we should like to emphasize that the proof requires but the slightest modification of methods used in [2], where the same is carried out for the zero level. The technical preliminaries, exploiting the interrelation of metric, symplectic and orbit structure, are indeed almost identical. We shall nevertheless give an outline here, in an attempt to be reasonably self-contained.

For $x \in M$ we provide a positive definite inner product for g^* which is invariant under the coadjoint action of $I(x, G)$ (this group is compact, since Φ is a proper action) and denote it by $(\, , \,)_x$. By abuse of notation # and b denote both the raising and lowering operations with respect to

$$g_x \;\; : T_xM \underset{\#}{\overset{b}{\rightleftarrows}} T_xM^* \qquad \text{and}$$

$$(\, , \,)_x : g \underset{\#}{\overset{b}{\rightleftarrows}} g^*.$$

Using $(\, , \,)_x$ the adjoint of $T_xJ : T_xM \longrightarrow g^*$ is defined as

$T_xJ^+ : g^* \longrightarrow T_xM$ by the condition:

$(T_xJ\, v, \mu)_x = g_x(v, T_xJ^+\mu)$ f.a. $v \in T_xM$.

It follows that

(1) $$T_xJ^+\mu = I_x\,[(\mu^\#)_M(x)]$$

and

(2) $$\operatorname{Im} T_xJ = [\ker T_xJ^+]^\perp.$$

We further need a characterization of M_H and its tangent space at a point $x_0 \in M_H$ in terms of the momentum mapping and h, the Lie algebra of H:

$$H \subset \hat{I}(x, G) \quad \text{iff} \quad \zeta_M(x) = 0 \quad \text{f.a.} \quad \zeta \in h$$

and, as the dimension of the isotropy group cannot increase locally, equality holds on the left hand side, in a neighbourhood of x_0.

Now $\zeta_M(x) = 0$ iff $\langle T_xJ\, v, \zeta\rangle = d\hat{J}(\zeta)_x \; v = 0$ f.a. $v \in T_xM$ and by linearization at x_0:

(3) $$T_{x_0}M_H = \{v \in T_xM/\; d^2J(\zeta)_{x_0}(u, v) = 0 \text{ f.a. } u \in T_{x_0}M, \zeta \in h\}\,(3).$$

The following properties of the bitangential map $d^2\hat{J}(\zeta)_{x_0}$ for $\zeta \in h$ will be used:

(3) The bitangential map of $\hat{J}(\zeta) : M \longrightarrow \mathbb{R}$ at x_0 is well defined since $d\hat{J}(\zeta)_{x_0}$ is the zero map for $\zeta \in h$ by ad*-equivariance.

(4) $[\xi,\zeta]=0 \Rightarrow \mathrm{d}^2\hat{J}(\zeta)_{x_0}(\xi_M(x_0),v)=0$ f.a. $v\in T_{x_0}M$

and I-invariance of $\mathrm{d}^2\hat{J}(\zeta)_{x_0}$:

(5) f.a. $u,v\in T_{x_0}M$: $\mathrm{d}^2\hat{J}(\zeta)_{x_0}(Iu,Iv)=\mathrm{d}^2\hat{J}(\zeta)_{x_0}(u,v)$.

Outline of proofs:
ad (4): $\hat{J}(\zeta)(\Phi_g x)=\hat{J}(\mathrm{Ad}^*_{g^{-1}}\zeta)(x)$ by Ad*-equivariance, differentiate w.r.t. x and at x_0 for $g=\exp(t\xi)$ w.r.t. t;
ad (5): $\mathrm{d}^2\hat{J}(\zeta)_{x_0}(u,v)=\mathbf{L}_u(\mathrm{d}\hat{J}(\zeta)(v))|_{x_0}=\mathbf{L}_u\omega_x(\zeta_M(x),v)|_{x_0}$ for any smooth extensions of u, v to vector fields; the desired result is obtained using $\zeta_M(x_0)=0$, I-invariance of ω and the identity $[Iu,\zeta_M]=I[u,\zeta_M]$, which follows from the fact that G acts by almost complex maps.

As a corollary from (3) and (5) one has that $T_{x_0}M_H$ is I-invariant and hence that for the inclusion map $i:M_H\longrightarrow M$, $i^*\omega$ is nondegenerate in a neighbourhood of x_0 in M_H : M_H is a symplectic submanifold of M.

After these preliminaries, we turn to the proof of (I); Let $\mu\in g^*$ and $H\subset G$ be fixed, $M_H\neq\phi$:

REGULARITY OF $J^{-1}(\mu)\cap M_H$, PROOF OF STATEMENT (I)

Since this is a local issue let $x_0\in J^{-1}(\mu)\cap M_H$ and use the $(\ ,\ )_{x_0}$-orthogonal decomposition $g^*=\mathrm{Im}\,T_{x_0}J\oplus\ker T_{x_0}J^+$ (cf. (2)) with the induced projection $\mathbb{P}:g^*\longrightarrow \mathrm{Im}\,T_{x_0}J$ to define the auxiliary map

$$\mathbb{P}\circ J: M\longrightarrow \mathrm{Im}\,T_{x_0}J.$$

We shall show that in a neighbourhood of x_0:

(i) $(\mathbb{P}\circ J)^{-1}(\mathbb{P}\mu)\cap M_H$ is a submanifold of M_H and
(ii) $(\mathbb{P}\circ J)^{-1}(\mathbb{P}\mu)\cap M_H=J^{-1}(\mu)\cap M_H$.

Together these establish (I).
ad (i):
Observe that by Ad*-equivariance of J:

(6)
$$J(M_H)\subset g^*_{x_0}:=\{\mu\in g^*/\mathrm{Ad}^*_{h^{-1}}\mu=\mu \quad \text{f.a.} \quad h\in H\}$$
$$=\{\xi\in g/\,[\xi,\zeta]=0 \quad \text{f.a.} \quad \zeta\in h\}^b.$$

Conversely:

(7) $T_{x_0}J^+(g^*_{x_0})\subset T_{x_0}M_H$.

(By (3), (7) is equivalent to

$$\mathrm{d}^2\hat{J}(\zeta)_{x_0}(T_{x_0}J^+\mu,v)=0 \quad \text{f.a.} \quad v\in T_{x_0}M,\ \mu\in g^*_{x_0},\ \zeta\in h$$

which in turn by (1) and (2) is equivalent to

$$d^2\hat{J}(\zeta)_{x_0}((\mu^{\#})_M(x_0), v) = 0 \quad \text{f.a.} \quad v \in T_{x_0}M,\ \mu \in g^*_{x_0},\ \zeta \in h$$

which follows from (4)).

Now by (6) $\mathbf{P} \circ J$ restricts to a map

$$\mathbf{P} \circ J|_{M_H} : M_H \longrightarrow \mathbf{P}\, g^*_{x_0},$$

which can be shown to be regular at x_0, i.e.

$$\mathbf{P} \circ T_{x_0}J|_{T_{x_0}M_H} : T_{x_0}M_H \longrightarrow \mathbf{P}\, g^*_{x_0} \qquad \text{is surjective.}$$

Proof. $\mathbf{P} = T_{x_0}J \circ [T_{x_0}J^+ \circ T_{x_0}J]^{-1} \circ T_{x_0}J^+$ where $T_{x_0}J^+ \circ T_{x_0}J$ is regarded as an automorphism of $\operatorname{Im} T_{x_0}J^+$ which furthermore leaves invariant $T_{x_0}M_H \cap \operatorname{Im} T_{x_0}J^+$ by (6) and (7). Let $\mu \in \mathbf{P}\, g^*_{x_0}$ i.e.

$$\begin{aligned} \mu = \mathbf{P}\mu = \mathbf{P}\nu \quad &\text{for some} \quad \nu \in g^*_{x_0} \\ = T_{x_0}J \circ [T_{x_0}J^+ \circ T_{x_0}J]^{-1} \circ\ &T_{x_0}J^+\nu \end{aligned}$$

$$\subset \in T_{x_0}M_H \supset$$

$$\subset\!\!\text{———————}\in T_{x_0}M_H\text{———————}\!\!\supset$$

$$\Rightarrow \mu \in \operatorname{Im} \mathbf{P} \circ T_{x_0}J|_{T_{x_0}M_H} \quad \text{f.a.} \quad \mu \in \mathbf{P}\, g^*_{x_0}. \qquad \blacksquare$$

Observe that one can also introduce a projection

$$\tilde{\mathbf{P}} : g^* \longrightarrow \operatorname{Im}\, [T_{x_0}J|_{T_{x_0}M_{(H)}}]$$

which agrees with $\mathbf{P}$ on $g^*_{x_0}$. This is possible since $\operatorname{Im}\, (T_{x_0}J|_{T_{x_0}M_{(H)}}) \supset$ $\supset \operatorname{Im}\, (T_{x_0}J|_{T_{x_0}M_H}) = \mathbf{P}\, g^*_{x_0}$. In restriction to M_H one still has

$$(8) \qquad (\tilde{\mathbf{P}} \circ J)^{-1}(\mathbf{P}\mu) \cap M_H = (\mathbf{P} \circ J)^{-1}(\mathbf{P}\mu) \cap M_H.$$

The first choice, however, is better adapted to the proof of (ii) because of the possible degeneracy of ω in $T_{x_0}M_{(H)}$.

ad (ii):

We observe that $\mathrm{id}_{g^*} - \mathbf{P}$ is locally constant on M_H: by Ad*-equivariance of J, for $x \in M_H$:

$$\begin{aligned} & d\hat{J}(\zeta)\, v = \langle T_xJ(v), \zeta\rangle = 0 \quad && \text{f.a.} \quad v \in T_xM, \quad \zeta \in h \\ \Longleftrightarrow \quad & (T_xJ(v), \zeta^b)_{x_0} = 0 && \text{f.a.} \quad v \in T_xM, \quad \zeta \in h \\ \Longleftrightarrow \quad & (T_xJ(v), \mu)_{x_0} = 0 && \text{f.a.} \quad v \in T_xM, \quad \mu \in h^b = \operatorname{Ker} T_{x_0}J^+ \quad (\text{cf. } (1)) \\ \text{or} \quad & T_x[(\mathrm{id}_{g^*} - \mathbf{P}) \circ J] = 0 && \text{for} \quad x \in M_H. \end{aligned}$$

From the definitions $J^{-1}(\mu) \cap M_H \subset (\mathbf{P} \circ J)^{-1}(\mathbf{P}\mu) \cap M_H$ and $\mathbf{P} \circ J$ is constant on $(\mathbf{P} \circ J)^{-1}(\mathbf{P}\mu) \cap M_H$. Hence J itself is constant on $(\mathbf{P} \circ J)^{-1}(\mathbf{P}\mu) \cap M_H$ and (ii) holds. ■

The extension from M_H to $M_{(H)}$ needed for (II) is almost trivial for the zero momentum level, by means of G-slices, locally at any point, since $J^{-1}(0)$ is G-invariant. In contrast one has to do with G_μ-invariance, not necessarily $G_\mu = G$, in the general case and therefore the orbit structure under the induced action of G_μ will have to be considered, too.

REGULARITY OF $J^{-1}(\mu) \cap M_{(H)}$, PROOF OF STATEMENT (II)

Let $x_0 \in J^{-1}(\mu) \cap M_{(H)}$, w.l.o.g. $\hat{I}(x_0, G) = H$. By Ad*-equivariance $H \subset G_\mu$. To take into account the orbit structure w.r.t. G_μ we consider $M_{(H)G_\mu}$, the submanifold of points whose orbit type for the action of G_μ is (H). Obviously $M_H \subset \subset M_{(H)G_\mu}$. Choosing an affine G_μ-slice at x_0 in $M_{(H)}$, L_{x_0} say, it is easily verified that $\tilde{L}_{x_0} := L_{x_0} \cap M_{(H)G_\mu}$ is an affine G_μ-slice at x_0 in $M_{(H)G_\mu}$. Furthermore $\tilde{L}_{x_0}$ is contained in M_H:

Let $y \in \tilde{L}_{x_0} \Rightarrow \hat{I}(y, G_\mu) \subset H$ by the definition of slices

$$\text{and} \quad \hat{I}(y, G_\mu) \sim H \quad \text{as} \quad y \in M_{(H)G_\mu}$$

$$\Rightarrow \hat{I}(y, G_\mu) = H \quad \text{and} \quad \hat{I}(y, G_\mu) = \hat{I}(y, G) \quad \text{as} \quad y \in M_{(H)}$$

$$\Rightarrow \tilde{L}_{x_0} \subset M_H.$$

Hence $J^{-1}(\mu) \cap M_H \cap \tilde{L}_{x_0} = J^{-1}(\mu) \cap \tilde{L}_{x_0}$ and $J^{-1}(\mu) \cap \tilde{L}_{x_0}$ is a submanifold locally, as at x_0 $J^{-1}(\mu) \cap M_H$ and $\tilde{L}_{x_0}$ are transverse in M_H:

$$T_{x_0}\tilde{L}_{x_0} = (T_{x_0} G_\mu . x_0)^\perp \cap T_{x_0} M_H$$

and (cf. the proof of I)

$$T_{x_0}(J^{-1}(\mu) \cap M_H) = \ker (\mathbf{P} \circ T_{x_0} J) \cap T_{x_0} M_H = \ker T_{x_0} J \cap T_{x_0} M_H.$$

On the other hand $J^{-1}(\mu) \cap L_{x_0} = J^{-1}(\mu) \cap \tilde{L}_{x_0}$:

Let $y \in L_{x_0} \cap J^{-1}(\mu) \Rightarrow \hat{I}(y, G) \subset G_\mu \Rightarrow \hat{I}(y, G) = \hat{I}(y, G_\mu) \Rightarrow \hat{I}(y, G_\mu) \sim H \Rightarrow \Rightarrow y \in \tilde{L}_{x_0}$.

Thus the regularity of $J^{-1}(\mu)$ in restriction to a G_μ-slice at $x_0 \in M_{(H)}$ is established and as pointed out above the extension of the result to a neighbourhood of x_0 in $M_{(H)}$ is simple, using the G_μ-invariance of both $J^{-1}(\mu)$ and $M_{(H)}$. An equivariant local diffeomorphism between $L_{x_0} \times G_\mu/H$ and $M_{(H)}$, as described below lemma (1), restricts to an equivariant embedding of $(J^{-1}(\mu) \cap L_{x_0}) \times G_\mu/H$ into $M_{(H)}$, whose image is $J^{-1}(\mu) \cap M_{(H)}$. The local structure of the manifold $J^{-1}(\mu) \cap M_{(H)}$ in a neighbourhood of an orbit $G_\mu . x_0$ (i.e. local w.r.t. the topology of M/G_μ) is that of an associated fibre bundle with standard fibre $J^{-1}(\mu) \cap L_{x_0}$ analogous to the statement of lemma (2).

We have in particular already found that the passage to the quotient $J^{-1}(\mu) \cap$ $\cap M_{(H)}/G_\mu$ is smooth, where $J^{-1}(\mu) \cap L_{x_0}$ provides a local representation of the manifold structure of the quotient. ■

INDUCED SYMPLECTIC STRUCTURE ON THE QUOTIENT MANIFOLD

Finally, to gain (III), we shall show that μ is a weakly regular value of the momentum mapping in restriction to $M_{(H)}$, and that here as in the standard case this criterion guarantees nondegeneracy of the induced 2-form on the quotient $J^{-1}(\mu) \cap M_{(H)}/G_\mu$, although the 2-form induced on $M_{(H)}$ may well be degenerate.

LEMMA (3). *Any $\mu \in g^*$ is a weakly regular value of $J|_{M_{(H)}}$.*

Proof. With the result of the previous paragraph it remains to show that $\ker T_{x_0} J \cap T_{x_0} M_{(H)} = T_{x_0}(J^{-1}(\mu) \cap M_{(H)})$ for $x_0 \in J^{-1}(\mu) \cap M_{(H)}$:

$$T_{x_0}(J^{-1}(\mu) \cap M_{(H)}) = T_{x_0} M_{(H)} \cap [T_{x_0}(M_{(H)} \cap L_{x_0}) + T_{x_0} G_\mu . x_0]$$
$$\subset T_{x_0} M_{(H)} \cap \ker T_{x_0} J \quad \text{by Ad*-equivariance.}$$

The converse inclusion proves to be much harder:
Making use of the local representation of $J^{-1}(\mu) \cap M_H$ as

$(\tilde{\mathbf{P}} \circ J)^{-1}(\mathbf{P}\mu) \cap M_H$, cf. (8), where $\tilde{\mathbf{P}} \circ J : M_{(H)} \longrightarrow \operatorname{Im} T_{x_0} J|_{T_{x_0} M_{(H)}}$

is regular by construction, we have:

$$T_{x_0}((\mathbf{P} \circ J)^{-1}(\mathbf{P}\mu) \cap M_{(H)}) = T_{x_0} M_{(H)} \cap \ker (\tilde{\mathbf{P}} \circ T_{x_0} J)$$

and hence

$$\begin{aligned} \ker T_{x_0} J \cap T_{x_0} M_{(H)} &\subset \ker (\tilde{\mathbf{P}} \circ T_{x_0} J) \cap T_{x_0} M_{(H)} \\ &= T_{x_0}((\tilde{\mathbf{P}} \circ J)^{-1}(\mathbf{P}\mu) \cap M_{(H)}) \\ &\subset T_{x_0}((\mathbf{P} \circ J)^{-1}(\mathbf{P}\mu) \cap M_H) + T_{x_0} G . x_0 \\ &= T_{x_0}(J^{-1}(\mu) \cap M_H) + T_{x_0} G . x_0 . \end{aligned} \tag{9}$$

Thus $v \in \ker T_{x_0} J \cap T_{x_0} M_{(H)}$ can be represented according to (9) as

$$v = v_1 + v_2, \quad v_1 \in T_{x_0}(J^{-1}(\mu) \cap M_H), \; v_2 \in T_{x_0} G. x_0.$$

It follows that $v_1 \in \ker T_{x_0} J$, $v_2 \in \ker T_{x_0} J \cap T_{x_0} G. x_0 = T_{x_0} G_\mu . x_0$, where the last identity is a consequence of Ad*-equivariance, again. Thus:

$$\ker T_{x_0} J \cap T_{x_0} M_{(H)} \subset T_{x_0}(J^{-1}(\mu) \cap M_H) + T_{x_0} G_\mu . x_0 = T_{x_0}(J^{-1}(\mu) \cap M_{(H)}).$$ ■

To complete our efforts, we still have to demonstrate the validity of the criterion of weak regularity for nondegeneracy of the induced 2-form on the quotient in our case – and indeed the standard proof is easily adapted. First however the result:

LEMMA (4). *For any of the submanifolds of constant orbit type, $M_{(H)}$, and any $\mu \in g^*$, the symplectic form ω on M induces a unique symplectic form $\tilde{\omega}$ on the quotient manifold $J^{-1}(\mu) \cap M_{(H)}/G_\mu$ according to*

$$p^* \tilde{\omega} = i^* \omega,$$

for $i : J^{-1}(\mu) \cap M_{(H)} \longrightarrow M$ the inclusion map, and p the natural projection of the quotient manifold.

Proof. $\tilde{\omega}$ is well defined as ω is invariant under the action of G_μ. The crucial point is nondegeneracy. Let $x_0 \in J^{-1}(\mu) \cap M_{(H)}$ and consider ω_{x_0} in restriction to $T_{x_0}(J^{-1}(\mu) \cap M_{(H)})$. It is easily verified that

$$T_{x_0} G. x_0 \subset T_{x_0} M_{(H)} \quad \text{and} \quad \ker T_{x_0} J \cap T_{x_0} M_{(H)}$$

are ω-orthogonal complements in $T_{x_0} M_{(H)}$. Hence the degeneracy space of ω_{x_0} in $T_{x_0}(J^{-1}(\mu) \cap M_{(H)})$, using lemma (3), is

$$T_{x_0} G. x_0 \cap T_{x_0} M_{(H)} \cap \ker T_{x_0} J = T_{x_0} M_{(H)} \cap T_{x_0} G_\mu . x_0$$

which is factored out in the passage to the quotient. ■

To put the results so far obtained into a more concise form, we state the following

THEOREM. *Let (M, ω, I, g) be an almost Kähler structure, Φ a proper action of a Lie group G on M by automorphisms of the almost Kähler structure with associated* Ad*-*equivariant momentum mapping J.*

For any orbit type (H), $H \subset G$, and any momentum value $\mu \in g^*$, the quotient manifold:

$$p : J^{-1}(\mu) \cap M_{(H)} \longrightarrow J^{-1}(\mu) \cap M_{(H)}/G_\mu$$

exists and carries a unique symplectic structure $\tilde{\omega}$ *induced according to:*

$$p^*\tilde{\omega} = i^*\omega \qquad J^{-1}(\mu) \cap M_{(H)} \begin{array}{l} \nearrow M, \omega \\ \searrow J^{-1}(\mu) \cap M_{(H)}/G_\mu, \tilde{\omega}. \end{array}$$

In terms of reduction of phase space with symmetry:
A phase space (M, ω) with a symmetry which is also a symmetry of an almost Kähler structure (M, ω, I, g) extending the symplectic structure of phase space and which admits an Ad*-equivariant momentum mapping, can – without any obstructions – be reduced according to the following scheme:

(I) decomposition of M into the submanifolds of constant orbit type w.r.t. the action of the symmetry group;

(II) subsequent decomposition into momentum level sets, combined with the usual passage to quotients w.r.t. the residual symmetry groups.

This scheme extends naturally to Hamiltonian systems of the given symmetry. The subsystems obtained in (I) are invariant under the Hamiltonian flow and the reduced phase spaces carry induced Hamiltonian structures whose relation to the original one is just as stated in the standard formulation of reduction.

It should be pointed out that although (II) is strictly analogous to the usual process of reduction, the standard procedure is in general not applicable to the subsystems arising in (I) as these need not be symplectic manifolds or Hamiltonian subsystems. In this sense the proposed reduction scheme may indeed give rise to reduced phase spaces or Hamiltonian systems which cannot be obtained using the standard method.

To conclude this paragraph, we state without proof the criterion for $M_{(H)}$ *to* be a symplectic submanifold of M locally, i.e. for ω_{x_0} to be nondegenerate in restriction to $T_{x_0}M_{(H)}$ at a point $x_0 \in M_{(H)}$: ω_{x_0} is nondegenerate on $T_{x_0}M_{(H)}$ iff $G_\mu \subset N(H)$, where $\mu = J(x_0)$ and $N(H)$ denotes the normalizer of H in G. Note this additional link between the momentum mapping and the local structure of the symmetry. The condition is open in M.

APPLICATION TO THE COTANGENT BUNDLE OF RIEMANNIAN MANIFOLDS

In this paragraph the applicability of the proposed reduction scheme is illustrated for an obvious and typical class of phase spaces with symmetry:
Cotangent bundles of Riemannian manifolds with symmetries induced by the

action of a group of isometries on the base manifold. The cotangent bundle with such lifted symmetries arises naturally in many applications and has long been an object of investigation in the study of reduction, [1]. Typical results include specific representations of reduced phase spaces, for the case that the base manifold itself is a principal fibre bundle w.r.t. the symmetry group (i.e. a most regular orbit structure on the base manifold is required) and for special values of the momentum map, see e.g. [7] and [11].

Here we shall require the symmetry to be induced by a proper action of a group of isometries on the base manifold but be quite general w.r.t. orbit structure and momentum values and concentrate on general applicability alone, of the above process of reduction in this natural case, without any further requirements.

Given the cotangent bundle of any Riemannian manifold we shall extend its canonical symplectic structure to an almost Kähler structure such that any symplectomorphism induced by an isometry of the base manifold becomes an automorphism of the almost Kähler structure (4).

First some convenient notation and technical details for dealing with connections on M:

Let for the following (M, g) be a Riemannian manifold. We regard TM and T^*M as fibre bundles associated with the bundle of linear frames, $L(M)$, and denote the respective projections as well as their tangential maps by π:

$$\pi : T^*M \longrightarrow M,$$

$$\pi : TM \longrightarrow M \quad \text{etc..}$$

Let a connection Γ on M be given in terms of the designation of horizontal subspaces in the associated bundles: for $\alpha_x \in T^*M$ e.g. (where the notation implies $\pi(\alpha_x) = x$) we denote by Q_{α_x} the horizontal subspace of $T_{\alpha_x} T^*M$, while K_{α_x} denotes the corresponding vertical subspace or tangent space to the fibre,

$$(10) \qquad T_{\alpha_x} T^*M = Q_{\alpha_x} \oplus K_{\alpha_x}.$$

We choose the same notation for TM.

A connection Γ gives rise to natural isomorphisms between the tangent spaces T_xM and the horizontal subspaces at points of the fibre $\pi^{-1}(x)$, the horizontal lift, cf. [6]. For the cotangent bundle, explicitly: the horizontal lift of $X \in T_xM$ to $\alpha_x \in \pi^{-1}(x)$, $X^H(\alpha_x)$ is defined by the requirements: $X^H(\alpha_x) \in Q_{\alpha_x}$ and $\pi(X^H(\alpha_x)) = X$.

(4) Observe that the Riemannian structure may in its turn be chosen so as to make a given action an action by isometries, e.g. in the case of a compact group.

Even without a connection there is a natural isomorphism between the vertical subspaces at any two points of the same fibre, corresponding to translation in the standard fibre of the vector bundle. This we shall call the vertical lift, which for TM in particular, gives rise to an isomorphism between $T_xM \simeq K_{0_x} \subset T_{0_x}TM$ and K_{v_x} for $v_x \in \pi^{-1}(x)$: $X \in T_xM$ is mapped to $X^V(v_x) \in K_{v_x}$ represented as $X^V(v_x) = \partial_t|_0 (v_x + tX)$. The inverse of this latter map will be denoted as $(X^V(v_x))\downarrow = X$. The same construction and notation applies to T^*M.

Horizontal and vertical lifts extend naturally to vector fields or sections of the bundle and give rise to horizontal and vertical lift vector fields, i.e. vector fields on the bundle which are related to vector fields on the base manifold by the respective lift.

We fix an – at first arbitrary – connection Γ on M. The canonical symplectic form on T^*M is given as $\omega_0 = -\,d\theta_0$, $\theta_0(\alpha_x) = \alpha_x \circ \pi$. We use the decomposition (10) to represent ω_0 in terms of horizontal and vertical components of the arguments:

LEMMA (5). *For $Z_i = X_i + Y_i \in T_{\alpha_x}T^*M$, $X_i \in Q_{\alpha_x}$, $Y_i \in K_{\alpha_x}$, $i = 1, 2$:*

$$\omega_0(\alpha_x)(Z_1, Z_2) = \langle Y_2\downarrow, \pi(X_1)\rangle - \langle Y_1\downarrow, \pi(x_2)\rangle,$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between T_xM^* and T_xM.

Outline of proof. Consider $\omega_0(X_1, X_2)$, $\omega_0(Y_1, Y_2)$ and $\omega_0(X_1, Y_2)$ separately and extend the arguments to vertical or horizontal lift vector fields, rsp.. Expressing $\omega_0 = -\,d\theta_0$ in terms of θ_0 and commutators of the arguments, the first two contributions are shown to vanish, while the mixed terms give the desired result. ∎

The intended almost complex structure on T^*M, I, can be given in restriction to horizontal and vertical arguments, rsp.. Let $\#$, b denote raising and lowering operations w.r.t. the metric g of the base manifold and define:

$$\text{(11)}\qquad \begin{aligned} I|_{Q_{\alpha_x}} &: X \longmapsto [\pi(X)^\flat]^V(\alpha_x) \\ I|_{K_{\alpha_x}} &: Y \longmapsto -[Y\downarrow^\#]^H(\alpha_x) \quad \text{for } \alpha_x \in T^*M,\ X \in Q_{\alpha_x},\ Y \in K_{\alpha_x}, \end{aligned}$$

i.e. I exchanges horizontal and vertical components, inserting a minus sign in one case. Note the dependence on the choice of connection.

LEMMA (6). *I as defined by (11) is an almost complex structure on T^*M which leaves ω_0 invariant.*

Outline of proof. Differentiability of I and the identity $I^2_{\alpha_x} = -\operatorname{id}_{T_{\alpha_x}T^*M}$ are obvious. Invariance of ω_0, i.e. $\omega_0(\alpha_x)(IZ_1, IZ_2) = \omega_0(\alpha_x)(Z_1, Z_2)$ is verified directly from the definitions using lemma (5).

Remark. For a torsion-free connection Γ, I as defined by (11) is a complex structure, i.e. its Nijenhuis-torsion vanishes [3], [6], iff the connection is flat.

From I-invariance of ω_0 it is obvious that $\hat{g}$, defined by

(12) $$\hat{g}(\alpha_x)(Z_1, Z_2) := \omega_0(\alpha_x)(Z_1, IZ_2)$$

for $Z_1, Z_2 \in T_{\alpha_x}T^*M$, is a hermitian pseudo-metric on T^*M.

LEMMA (7). *$\hat{g}$ as defined by (12) is a Riemannian metric on T^*M.*

Hence $(T^*M, \omega_0, I, \hat{g})$ is an almost Kähler structure whose fundamental 2-form is equal to the canonical symplectic form. (This holds for any connection Γ used in (11)).

In fact $\hat{g}$ turns out to be the natural extension of g to T^*M by means of the connection Γ: horizontal and vertical subspaces are orthogonal complements w.r.t. $\hat{g}$ and

(13)
$$\hat{g}(\alpha_x)(X_1, X_2) = g_x(\pi(X_1), \pi(x_2)) \quad \text{for} \quad X_1, X_2 \text{ horizontal},$$
$$\hat{g}(\alpha_x)(Y_1, Y_2) = g_x(Y_1\downarrow^\#, Y_2\downarrow^\#) \quad \text{for} \quad Y_1, Y_2 \text{ vertical}.$$

Let now $\Phi : G \times M \longrightarrow M$ be an action by isometries on M. Φ is canonically lifted to T^*M [1]:

$$\Phi^{T^*} : G \times T^*M \longrightarrow T^*M,$$
$$(g, \alpha_x) \longmapsto (T\Phi_{g^{-1}})^* \alpha_x,$$

to give an action by symplectomorphisms on T^*M. The action is in fact by automorphisms of the proposed almost Kähler structure if we choose Γ to be the Levi-Civita connection for (M, g). As pointed out above it suffices to show that Φ^{T^*} preserves $\hat{g}$ in addition to ω_0. This is obvious from (13) since $T\Phi$ is an isometry w.r.t g and in particular preserves horizontal and vertical subspaces of the Levi-Civita connection. Summing up the results we have established the following

PROPOSITION. *Let (M, g) be a Riemannian manifold and $\Phi : G \times M \longrightarrow M$ a proper action by isometries.*

The induced action of G on T^*M is a symmetry of a suitably chosen almost Kähler structure whose fundamental 2-form is the canonical symplectic form of T^*M.

Since the existence of an Ad*-equivariant momentum mapping associated with a symmetry of this kind is guaranteed [1], the proposed scheme of reduction is generally applicable to this class of phase spaces with symmetry.

CONCLUSIONS

Apart from the appealing generality in the formulation of the proposed scheme of reduction for almost Kähler symmetric systems, which is due to its universal applicability within the range of such systems, it is to be expected that it facilitates the analysis not only of all the systems which are naturally endowed with an almost Kähler structure, but of those which permit the construction of an auxiliary almost Kähler structure.

Our construction for the cotangent bundle, in particular, makes contact with many of the classical applications of reduction and allows for a general treatment of geodesic flow systems along these lines. The derivation of the central result, however, also provides some further insignt into the nature of the singularities of momentum level sets. From the regularity results obtained, all momentum level sets are seen to be decomposed into their regular components in a reduction w.r.t. orbit type or symmetry type, extending the results in [2]. In this sense our analysis also points at a possible approach for dealing with a particularly accessible class of singularities or bifurcations in mechanical systems.

REMARK ON THE REQUIREMENT OF PROPER ACTIONS

The full group of isometries of a Riemannian structure (with a finite number of connected components), $\mathscr{I}$, is a Lie group w.r.t. the compact open topology and induces a proper action on the manifold [5]. Obviously the full group of automorphisms of an almost Kähler structure is an algebraic subgroup of $\mathscr{I}$. By continuity the full automorphism group of an almost Kähler structure is closed in $\mathscr{I}$, hence a Lie group with proper action on the manifold. Similarly, any action of a Lie group G by almost Kähler automorphisms gives rise to a proper action of the closure of G in $\mathscr{I}$ by almost Kähler automorphisms.

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REFERENCES

[1] R. ABRAHAM, J.E. MARSDEN: *Foundations of Mechanics*, second edition. The Benjamin/Cummings Publ. Comp., Reading 1978.

[2] J. ARMS, J.E. MARSDEN, V. MONCRIEF: *Symmetry and Bifurcation of Momentum Mappings*, in: Comm. Math. Phys. 78 (1981).

[3] S. CHERN: *Complex Manifolds Without Potential Theory*, Springer, New York 1979.

[4] R. HERMANN: *Differential Geometry and the Calculus of Variations*, second edition. Math. Sci. Press, New York 1977.

[5] S. KOBAYASHI: *Transformation Groups in Differential Geometry*, Springer, Berlin, Heidelberg 1972.

[6] S. KOBAYASHI, K. NOMIZU: *Foundations of Differential Geometry*, John Wiley & Sons, New York 1963.

[7] M. KUMMER: *On the Construction of the Reduced Phase Space of a Hamiltonian System with Symmetry*, in: Indiana Univ. Math. Journal **30** (1981).

[8] J.E. MARSDEN, A. WEINSTEIN: *Reduction of Symplectic Manifolds with Symmetry*, in: Reports in Math. Phys. **5** (1974).

[9] J. MOSER: *Various Aspects of Integrable Hamiltonian Systems*, in: J. Guckenheimer, J. Moser, S.E. Newhouse: *Dynamical Systems*, CIME Lectures Bressanone Italy, June 78. Birkhäuser, Boston 1980.

[10] R.S. PALAIS: *The Classification of G-Spaces*, in: Memoirs of the AMS **36** (1960).

[11] A. WEINSTEIN: *A Universal Phase Space for Particles in Yang-Mills Fields*, in: Lett. Math. Phys. **2** (1978).

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